3.16.1. Derived Rule Problems

For each of the following deductive systems, show that one or more of our deductive rules are **derived rules** in that system (by constructing a deduction of that rule, in that system).

1. Show that the rule **MP** can be treated as a derived rule in the Chapter Three deductive system, by providing a deduction of the argument "$(P \rightarrow Q)$. P $\therefore$ Q" that doesn't use MP.

2. Show that the rule **MT** can be treated as a derived rule in the Chapter Three deductive system, by providing a deduction of the argument "$(P \rightarrow Q)$. ~Q $\therefore$ ~P" that doesn't use MT.

3. Show that the rule **R** can be treated as a derived rule in the Chapter Three deductive system, by providing a deduction of the argument "P $\therefore$ P" **using only CD and MP**. *(From Kalish, Montague, and Mar 1980: 48 #38)*

4. Show that the rule ~– can be treated as a derived rule in the Chapter Three deductive system, by providing a deduction of the argument "~~P $\therefore$ P" **using only ID, CD, and MP**. *(From Kalish, Montague, and Mar 1980: 48 #38)*

5. The deductive system **DS1** has only ID, plus the the rules ∧−, ∧+, ∨−, ∨+, and “Negated Conditional” (“~→”).

Negated Conditional (~→)

$$\frac{\sim(\bullet \rightarrow \blacktriangle)}{\therefore\ (\bullet \wedge \sim\blacktriangle)} \qquad \frac{(\bullet \wedge \sim\blacktriangle)}{\therefore\ \sim(\bullet \rightarrow \blacktriangle)}$$

Thanks to ~→, any **CD** from our system of deduction can be converted into an ID in DS1.[1]

5a. Show that **MT** is a derived rule in DS1.

5b. Show that **MP** is a derived rule in DS1.

6. The deductive system **DS2** is like our Chapter Three deductive system except that it lacks the rule ∨− and instead has the rule **Separation of Cases (SC)**.

Separation of Cases (SC)

$$\frac{\begin{array}{c}(\bullet \vee \heartsuit) \\ (\bullet \rightarrow \blacktriangle) \\ (\heartsuit \rightarrow \blacktriangle)\end{array}}{\blacktriangle}$$

[1] Because DS1 has **ID** and the rules ∨ – and ∨+, the rules R, ~–, and ~+ are already derived rules in this system – as shown in 2.40.1 Problems 2, 3, and 5.

So the following argument is an example of (SC).

$$\begin{array}{c} 1.\ (P \vee Q) \\ 2.\ (P \rightarrow R) \\ 3.\ (Q \rightarrow R) \\ \hline \therefore\ R \end{array}$$

6a. Show that the above argument is deducible in the Chapter Three deductive system (and hence that **SC** can act as a derived rule in the Chapter Three system).

6b. Show that the following argument is deducible using only **CD**, **ID**, **R**, and **SC** (and hence that $\vee-$ is a derived rule in **DS2**).

$$\begin{array}{c} 1.\ (P \vee Q) \\ 2.\ \sim P \\ \hline \therefore\ Q \end{array}$$

6c. Show that the following argument is deducible using only **CD**, **ID**, $\wedge-$, **R**, and **SC**.

$$\begin{array}{c} 1.\ (\sim P \wedge \sim Q) \\ \hline \therefore\ \sim(P \vee Q) \end{array}$$

7. The following argument is an instance of the rule **Double Disjunction** (DD), discussed in Chapter Two.[2]

1. $(P \vee Q)$
2. $(P \vee {\sim}Q)$

$\therefore$ P

Provide a deduction of the above argument using only **~+**, **∨–**, **SC**, and **CD**.

8. Note that the argument "$(P \rightarrow (P \rightarrow Q))$ $\therefore$ $(P \rightarrow Q)$" is valid. We can use the rule Separation of Cases to **explain** the validity of this rule **deductively**: from the sentence "$(P \rightarrow (P \rightarrow Q))$," when accompanied by the sentences "$(P \vee {\sim}P)$" and "$({\sim}P \rightarrow (P \rightarrow Q))$," the conclusion "$(P \rightarrow Q)$" follows as an instance of Separation of Cases.

1. $(P \vee {\sim}P)$
2. **$(P \rightarrow (P \rightarrow Q))$**
3. $({\sim}P \rightarrow (P \rightarrow Q))$

$\therefore$ **$(P \rightarrow Q)$** 1, 2, 3, SC

But the other two premises, "$(P \vee {\sim}P)$" and "$({\sim}P \rightarrow (P \rightarrow Q))$," are **theorems** provable without premises[3] – in effect, built in the deductive system. In that sense, from the deductive system alone "$(P \rightarrow (P \rightarrow Q))$" entails "$(P \rightarrow Q)$" by Separation of Cases.

For each of the following arguments, use **Separation of Cases** plus theorems to deductively explain the validity of that argument.

8a. $(P \vee Q)$. $(P \rightarrow Q)$ $\therefore$ Q

8b. $({\sim}P \rightarrow P)$ $\therefore$ P

8c. $(P \rightarrow {\sim}(P \wedge Q))$ $\therefore$ ${\sim}(P \wedge Q)$

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[2] In 2.40.1 Problem 8

[3] "$(P \vee {\sim}P)$" is theorem **T2.2** from 2.41.1, and "$({\sim}P \rightarrow (P \rightarrow Q))$" is theorem **T3.3b** from 3.13.1 C.

8d. $(P \rightarrow (Q \rightarrow \sim P)) \therefore (Q \rightarrow \sim P)$

8e. $((P \rightarrow Q) \rightarrow P) \therefore P$

9. We can derive a number of rules using the rule **Contradictory Consequents (CC)**.

Contradictory Consequents (CC)

$$\frac{\begin{array}{c}(\bullet \rightarrow \blacktriangle)\\ (\bullet \rightarrow \sim\blacktriangle)\end{array}}{\therefore\ \sim\bullet}$$

For instance, MT becomes a derived rule (using only CD, R, and CC).

1.	$(P \rightarrow Q)$	
2.	$\sim Q$	
		Get: $\sim P$
		Get: $(P \rightarrow \sim Q)$
3.	P	ACD
4.	$\sim Q$	2, R
5.	$(P \rightarrow \sim Q)$	3, 4, CD
6.	$\sim P$	1, 5, CC

Provide **deductions** of each of the following arguments (using only the rules and forms of deduction specified).

9a. $P \therefore \sim\sim P$ (using only CD, R, and CC)

9b. $(P \rightarrow Q) . P \therefore Q$ (using only ID, CD, R, and CC)

10. We can derive a number of rules using a variant of the **Contradictory Consequents rule**, **(CC*)**.

Contradictory Consequents* (CC*)[4]

$$\frac{\begin{array}{c}(\sim\bullet \rightarrow \blacktriangle)\\(\sim\bullet \rightarrow \sim\blacktriangle)\end{array}}{\therefore \bullet}$$

Provide **deductions** of each of the following arguments (using only the rules and forms of deduction specified).

10a. ~~P ∴ P (using only CD, R, and CC*)

10b. (P → Q) . P ∴ Q (using only CD, R, MT, and CC*)

10c. (P → Q) . P ∴ Q (using only CD, R, CC, and CC*)

10d. (~P → ~Q) . P ∴ Q (using only CD, R, and CC*)

[4] This is the conditional analogue of the Chapter Two derived rule **Double Disjunction** (discussed in 2.40.1 Problem 8).

11. We can derive a number of rules using a **variant of Modus Tollens**, **(MT*)**.

Modus Tollens* (MT*)

$$\frac{\begin{array}{c}(\sim\bullet \to \sim\blacktriangle)\\ \blacktriangle\end{array}}{\therefore \bullet}$$

Provide **deductions** of each of the following arguments (using only the rules and forms of deduction specified). *(11a-11c adapted from Kalish, Montague, and Mar 1980: 44 #39)*

11a. P ∴ P (using only CD and MT*)

11b. ~~P ∴ P (using only CD and MT*)

11c. P ∴ ~~P (using only CD and MT*)

11d. (~P → ~Q) . P ∴ Q (using only MP, ~+, and the following rule of **Contraposition**)

Contraposition (CP)

$$\frac{(\sim\bullet \to \sim\blacktriangle)}{\therefore (\blacktriangle \to \bullet)}$$

11e. (~P → ~Q) ∴ (Q → P) (using only CD, R, and CC*)

11f. (Q → P) ∴ (~P → ~Q) (using only CD, R, and CC)